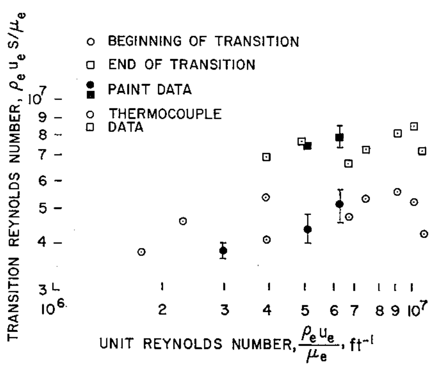

Fig. 2 Comparison of transition Reynolds numbers on 5° half-angle cones at $M_\infty = 7.4$.

location of the beginning and end of transition are also approximately the same. Similar paint data taken at other total pressures and transition Reynolds numbers are compared on Fig. 2.

The beginning of transition is defined as the intersection of straight lines faired through the laminar and transitional portions of the heating data. Similarly, the intersection of lines through the transitional and turbulent data defines the end of transition. Reynolds numbers based on surface distances to these intersections and boundary-layer-edge conditions are shown in Fig. 2 as a function of unit Reynolds number. Transition Reynolds numbers obtained using both paint and thermocouples compare very well and indicate that the paint technique can provide an adequate indication of boundary-layer transition.

References

[1] Jones, R. A. and Hunt, J. L., "Use of Fusible Temperature Indicators for Obtaining Quantitative Aerodynamic Heat-Transfer Data," TR-R-230, 1966, NASA.

[2] Stainback, P. C., "Some Effects of Roughness and Variable Entropy on Transition at a Mach Number of 8," *AIAA Journal*, Vol. 8, No. 1, Jan. 1970, pp. 173–175.

[3] Mateer, G. G. and Larson, H. K., "Unusual Boundary-Layer Transition Results on Cones in Hypersonic Flow," *AIAA Journal*, Vol. 7, No. 4, April 1969, pp. 660–664.

Optimum Relaxation Time for a Maxwell Core during Forced Vibration of a Rocket Assembly

BERNARD W. SHAFFER*
New York University, New York, N. Y.

AND

ROBERT I. SANN†
Ingersoll Rand Research Center, Princeton, N. J.

WHEN the core of a case-bonded viscoelastic assembly is made of a Maxwell solid, an optimum relaxation time is found, which minimizes the displacement amplitude and the bond-stress response at resonance. For a Voigt solid the displacement amplitude and the bond-stress response at resonance decreases with retardation time, but no optimum time exists in the same sense.

In a previous paper concerned with the forced vibration response of a case-bonded viscoelastic cylinder,[1] the authors presented numerical results that indicated the existence of an optimum relaxation time τ for a Maxwell solid. At the optimum relaxation time τ the bond stress amplitude response is minimized. It was observed, on the basis of some numerical calculations, that the optimum relaxation time τ decreases with increasing values of the resonant frequency ω.

It is the purpose of this brief Note to prove the existence of an optimum relaxation time τ for an assembly consisting of a solid cylinder bonded to a thin casing, if the cylinder is made of a material which is a Maxwell solid. It is also demonstrated that a Voigt solid has no optimum retardation time.

The present analysis starts with the law of conservation of energy for the system. Neglecting thermodynamic effects, the law of conservation of energy may be written

$$P_{\text{ext}} = (d/dt)(KE) + \int_V \sigma_{ij}\dot{\epsilon}_{ij}dV \tag{1}$$

where σ_{ij} are the components of stress and $\dot{\epsilon}_{ij}$ are the components of strain rate; t is the parameter time. The term P_{ext} is the rate of work of the external forces, and KE is the kinetic energy of the system. The integral represents the rate of work done by the internal stresses during deformation.

There is no change in KE over one cycle of sinusoidal vibration. Hence, the integral of Eq. (1) from $t = 0$ to $t = 2\pi/\omega$ may be written

$$\int_0^{2\pi/\omega} P_{\text{ext}}dt = \int_0^{2\pi/\omega}\int_V \sigma_{ij}\dot{\epsilon}_{ij}dVdt \tag{2}$$

In the particular problem under consideration, which is the same as that previously studied,[1] the rate of work of the external force may be written

$$P_{\text{ext}} = \int_{-\pi}^{+\pi} -pa\frac{\partial}{\partial t}u(a,\theta,t)d\theta \tag{3}$$

where p is the normal surface traction applied to the outer surface of the casing, u is the radial displacement of particles under load, and a is the radius of the common surface between the cylinder and its casing.

It is convenient to separate the stress and strain-rate components into deviatoric components S_{ij}, e_{ij}, and its mean normal components σ,ϵ, respectively. If this is done, the product $\sigma_{ij}\dot{\epsilon}_{ij}$ can be written

$$\sigma_{ij}\dot{\epsilon}_{ij} = S_{ij}\dot{e}_{ij} + 3\sigma\dot{\epsilon} \tag{4}$$

Should be assumption again be made[1] that the cylinder is elastic in dilatation

$$\sigma = K\epsilon \tag{5}$$

where K is the bulk modulus of elasticity. The integral on the right side of Eq. (2) then becomes

$$\int_0^{2\pi/\omega}\int_V \sigma_{ij}\dot{\epsilon}_{ij}dVdt = \int_0^{2\pi/\omega}\int_V S_{ij}\dot{e}_{ij}dVdt \tag{6}$$

Equation (6) shows that all vibratory energy dissipation is due to distortion and none to volume change.

Let us write the pressure load $p(\theta,t)$ as a phasor

$$p = Re(P_0e^{j\omega t}) \tag{7}$$

where P_0 is the complex amplitude and ω the real frequency. Then the radial displacement $u(r,\theta,t)$ can be expressed

$$u = Re[(U_0/P_0)(j\omega,r)P_0e^{j\omega t}] \tag{8}$$

where the complex displacement transfer function $U_0(j\omega,r)/P_0$ is given explicitly by Eq. (37a) of Ref. 1; namely

$$\frac{U_0}{P_0} = \frac{aJ_1(\alpha r)}{[\rho a\omega^2 + 2G - hE/a(1-\nu^2)]J_1(z) - (K + \frac{4}{3}G)zJ_0(z)} \tag{9}$$

Received July 16, 1970. The results presented in this paper were obtained in the course of research sponsored by NASA under Grant No. NGR-33-016-067 to New York University. The authors wish to thank H. Yanowitz of the Perkin-Elmer Corporation for programing the example discussed in this report.

* Professor of Mechanical Engineering. Associate Fellow AIAA.

† Chief, Solid Mechanics Research. Member AIAA.

where E,ν are, respectively, Young's modulus and Poisson's ratio for the casing material ρ is the mass density of the casing, per unit area of the middle surface; h is the wall thickness of the casing; and G is the shear modulus of the core. The terms $J_n(x)$ is Bessel's function of the order n and

$$\alpha = \omega[\gamma/(K + \tfrac{4}{3}G)]^{1/2} \qquad z = \alpha a \tag{10}$$

Substituting Eqs. (7) and (8) into Eq. (3), one finds that

$$P_{\text{ext}} = -2\pi a Re(P_0 e^{j\omega t}) Re[j\omega(U_0/P_0)(j\omega,a)P_0 e^{j\omega t}] \tag{11}$$

Then as a consequence of the lemma

$$\int_0^{2\pi/\omega} Re(ce^{j\omega t})Re(de^{j\omega t})dt = \frac{\pi}{\omega} Re(c\bar{d}) \tag{12}$$

it is found by substituting Eq. (9) into the left side of Eq. (2), the expression for the work of the external forces in one cycle of vibration becomes

$$\int_0^{2\pi/\omega} P_{\text{ext}}dt = 2\pi^2 a|P_0|^2 I_m\left[\frac{U_0}{P_0}(j\omega,a)\right] \tag{13}$$

In order to evaluate the double integral on the right side of Eq. (6), one writes the principal deviatoric strain components e_{rr}, $e_{\theta\theta}$, and e_{zz} as phasors,

$$e_{rr} = Re(E_1 e^{j\omega t}),\ e_{\theta\theta} = Re(E_2 e^{j\omega t}),\ e_{zz} = Re(E_3 e^{j\omega t}) \tag{14}$$

where E_1, E_2, and E_3 are the complex deviatoric strain amplitudes.

The principal deviatoric stress components S_{rr}, $S_{\theta\theta}$ and S_{zz} may be obtained using Eq. (14) and the definition of complex shear modulus $G_c(j\omega)$

$$S_{rr} = Re(2G_cE_1e^{j\omega t}),\ S_{\theta\theta} = Re(2G_cE_2e^{j\omega t}),\ S_{zz} = Re(2G_cE_3e^{j\omega t}) \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (6) and using the lemma of Eq. (12), one obtains for the work done by the internal stresses in one cycle of vibration the expression

$$\int_0^{2\pi/\omega}\int_V \sigma_{ij}\dot{\epsilon}_{ij}dVdt = 2\pi I_m[G_c(j\omega)] \times \int_A (|E_1|^2 + |E_2|^2 + |E_3|^2)dA \tag{16}$$

Equation (2) may now be applied by using the results of Eqs. (13) and (16). A relationship is obtained between U_0/P_0 and G_c involving the strain amplitudes; namely

$$I_m\left[\frac{U_0}{P_0}(j\omega,a)\right] = \frac{I_m[G_c(j\omega)]}{\pi a|P_0|^2}\int_A (|E_1|^2 + |E_2|^2 + |E_3|^2)dA \tag{17}$$

The complex, deviatoric strain amplitudes E_1, E_2, and E_3 are related to the displacement transfer function $(U_0/P_0)(j\omega,a)$ through the strain-displacement equations of Ref. 2; namely

$$E_1 = (U_0/P_0)(j\omega,a)[\tfrac{2}{3}\alpha J_0(\alpha r) - J_1(\alpha r)/r]P_0/J_1(z) \tag{18a}$$

$$E_2 = (U_0/P_0)(j\omega,a)\left[-\frac{\alpha}{3}J_0(\alpha r) + J_1(\alpha r)/r\right]P_0/J_1(z) \tag{18b}$$

$$E_3 = (U_0/P_0)(j\omega,a)\left[-\frac{\alpha}{3}J_0(\alpha r)\right]P_0/J_1(z) \tag{18c}$$

The area integral appearing on the right side of Eq. (17) can now be expressed as

$$\int_A (|E_1|^2 + |E_2|^2 + |E_3|^2)dA = 2\pi|P_0|^2\left|\frac{U_0(j\omega,a)}{P_0}\right|^2 m(z) \tag{19}$$

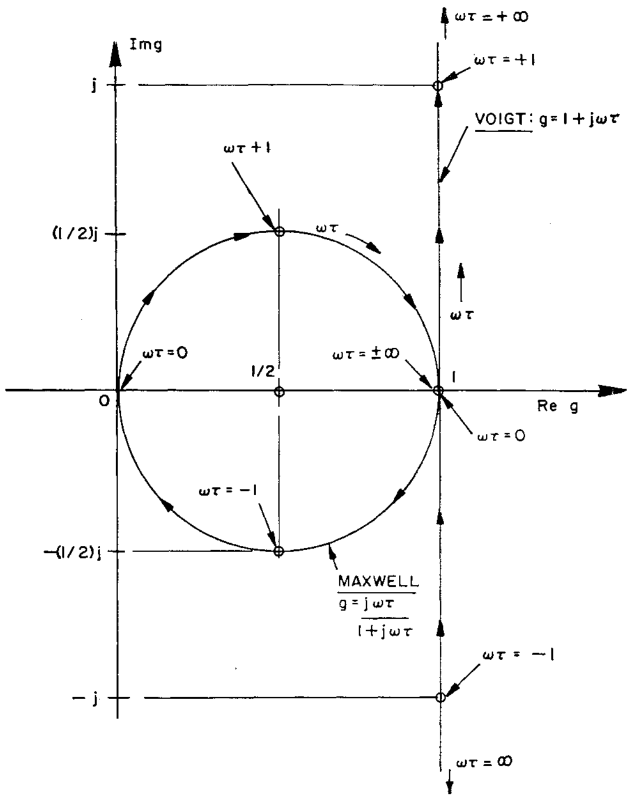

Fig. 1 Frequency loci of complex moduli for Voigt and Maxwell solids; $g = G_c(j\omega)/Go$.

The factor

$$m(z) = \frac{e^{-2j<z}}{|J_1(z)|^2}\int_0^z\left[\left|\frac{2}{3}J_0(x) - \frac{J_1(x)}{x}\right|^2 + \left|\frac{J_1(x)}{x} - \frac{1}{3}J_0(x)\right|^2 + \left|\frac{1}{3}J_0(x)\right|^2\right]xdx \tag{20}$$

is a non-negative real valued function of z involving integrals of Bessel functions.

When Eq. (19) is inserted into Eq. (17) one obtains a single expression in $U_0(j\omega,a)/P_0$ and $G_c(j\omega)$ alone; namely

$$I_m\left[\frac{U_0}{P_0}(j\omega,a)\right] = \frac{2}{a}m(z)\left|\frac{U_0}{P_0}(j\omega,a)\right|^2 I_m[G_c(j\omega)] \tag{21}$$

For the particular values of the parameters used in Ref. 1, $m(z)$ is practically insensitive to τ over a wide range of frequencies and time constants, for both Voigt and Maxwell solids. The term $m(z)$, does vary significantly with frequency.

In lightly damped systems the resonant frequencies are approximately equal to the respective resonant frequencies of the corresponding perfectly elastic system. Within such lightly damped systems, therefore, it follows that U_0/P_0 is approximately pure imaginary at resonance. The right-hand side of Eq. (21) is then non-negative, for positive ω, in which case U_0 must lead P_0 by approximately 90°. Therefore, it is reasonable to substitute $|U_0/P_0|$ for $I_m(U_0/P_0)$ and write Eq. (21) as

$$|U(j\omega,a)/P_0| = (a/2m)/I_m[G_c(j\omega)] \tag{22}$$

The coefficient $a/2m$ of Eq. (22) is practically independent of viscoelastic time constant, for the parameters considered.

In the case of Voigt and Maxwell solids, the complex shear modulus G_c depends on the time parameter τ only through the product $\omega\tau$, so that

$$G_c = G_0 g(j\omega\tau) \tag{23}$$

Table 1 Optimum relaxation time for Maxwell core material

	First	Second	Third
$\tau C_2^{(0)}/a$	0.012061	0.006540	0.004257
$\omega a/C_1^{(0)}$	2.729058	5.032956	7.735903
$\min\limits_{\tau C_2^{(0)}/a}\left[\max\limits_{\omega a/C_1^{(0)}}\left\lvert\frac{\Sigma_0^{(rr)}}{P_0}(j\omega,a)\right\rvert\right]$	501.268	242.630	179.112
$\omega\tau$	0.998985	0.999003	0.999387

where

$$g = 1 + j\omega\tau,\ \text{Voigt solid} \tag{24a}$$

$$g = j\omega\tau/(1 + j\omega\tau),\ \text{Maxwell solid} \tag{24b}$$

and the static modulus of rigidity G_0 is real.

The g loci are plotted in Fig. 1 as a function of $\omega\tau$. It is apparent that $Im\ g$ is unbounded for a Voigt solid, but has a maximum at $\omega\tau = 1$ for a Maxwell solid. Hence $1/Img$ is a monotonically decreasing expression for a Voigt solid, but first decreases, then increases for a Maxwell solid when $1/Img$ is regarded as a function of τ for a fixed positive value of ω.

Thus, there exists an optimum relaxation time τ, for a Maxwell solid, which minimizes the displacement amplitude at resonance. Moreover the optimum τ is simply the reciprocal of the resonant frequency ω, i.e.,

$$\tau_{\text{opt}} = 1/\omega,\ \text{Maxwell solid} \tag{25}$$

For a Voigt solid, the displacement amplitude at resonance decreases monotonically with increasing numerical values of τ. Thus, there is no optimum retardation time for a Voigt solid, in the sense previously stated.

At the same values of $\omega\tau$ and G_0 the Voigt solid always provides greater vibration attenuation than the Maxwell solid, since the former has a greater value of $|ImG_c|$.

Let us now examine the amplitude of the radial bond stress $\sigma_{rr}(a,\theta,t)$ at resonance. The appropriate stress transfer function is $[\Sigma_0^{(rr)}/P_0(j\omega,a)$ at the lowest circumferential wave number and is given by Eq. (39) of Ref. 1 as

$$\frac{\Sigma_0^{(rr)}(j\omega,a)}{P_0(j\omega)} = \frac{zJ_0(z) - 2(C_2/C_1)^2J_1(z)}{\left\{\frac{\rho}{h\gamma}\left(\frac{h}{a}\right)z^2 + \left(\frac{C_2}{C_1}\right)^2\left[2 - \frac{E}{G_c(1-\nu^2)}\left(\frac{h}{a}\right)\right]\right\}J_1(a) - zJ_0(z)} \tag{26}$$

where the dimensionless frequency

$$z = \omega a/C_1 \tag{27}$$

and the terms

$$C_1 = [(K + \tfrac{4}{3}G_c)/\gamma]^{1/2};\quad C_2 = [G_c/\gamma]^{1/2} \tag{28}$$

It may be related to the bond displacement transfer function $[U_0(j\omega,a)/P_0]$ by means of Eq. (9), thereby providing the equality

$$\frac{\Sigma_0^{(rr)}}{P_0}(j\omega,a) = d\left[\frac{U_0\,(j\omega,a)}{P_0}\right] \tag{29}$$

where the parameter d is defined by the relation

$$d = (\gamma/a)(C_1^2zJ_0(z)/J_1(z) - 2C_2^2) \tag{30}$$

For the numerical values used in Ref. 1; namely

$$\rho/h\gamma = 1.96;\quad E/G_0(1-\nu^2) = 22{,}500,$$
$$C_1^{(0)}/C_2^{(0)} = 30.35;\quad h/a = 0.1 \tag{31}$$

the parameter d is practically insensitive to τ over a wide range of frequencies ω for both Voigt and Maxwell materials.

Substitution of Eq. (29) into Eq. (22) yields the following result

$$\left|\frac{\Sigma_0^{(rr)}(j\omega,a)}{P_0}\right| = \left(\frac{ad}{2m}\right)\Big/ I_m[G_c(j\omega)],\ \text{at resonance} \tag{32}$$

The coefficient $ad/2m$ which appears in Eq. (31) is practically independent of the viscoelastic time constant τ. Thus, the same conclusions regarding optimum values of τ previously deduced for $|U_0/P_0|$ also hold with respect to $|\Sigma_0^{(rr)}/P_0|$. In particular τ given by Eq. (25) also minimizes $|\Sigma_0^{(rr)}/P_0|$ at resonance for a Maxwell solid.

In order to study numerically the degree of approximation inherent in Eq. (25), a digital computer program was developed to search for the optimum value of τ by iteration. The program was applied to the Maxwell cylinder assembly defined by Eq. (31). The program works in the following way: first, the parameter $\tau C_2^{(0)}/a$ is fixed and the parameter $\omega a/C_1^{(0)}$ is varied to determine a maximum value of $|\Sigma_0^{(rr)}/P_0|$. This locates the resonant frequency and peak amplitude as a function of relaxation time. Next $\tau C_2^{(0)}/a$ is varied and the resonant peak amplitudes so generated are examined for a minimum peak amplitude. An optimum $\tau C_2^{(0)}/a$ and corresponding resonant $\omega a/C_1^{(0)}$ are also determined during this process.

The calculation procedure described previously is performed for the first three resonant frequencies given in Ref. 1. Results are listed in Table 1.

It is apparent from the Table that the optimum relaxation time can be computed quite accurately from Eq. (25), for the particular example studied. The large stress amplitude ratios for this configuration attest to the low degree of damping in the system, and therefore to the validity of the assumption that $(U_0/P_0)(j\omega,a)$ is pure imaginary.

A comparison of the preceding $\omega a/C_1^{(0)}$ values with those given in Figures 2–4 of Ref. 2 for the corresponding all-elastic assembly, reveals that even at optimum damping, damping has negligible effect on resonant frequency.

References

[1] Shaffer, B. W. and Sann, R. I., "Forced Transverse Vibration of a Solid Viscoelastic Cylinder Bonded to a Thin Casing," *Journal of Applied Mechanics*, Vol. 36, Dec. 1969, pp. 827–834.

[2] Sann, R. I. and Shaffer, B. W., "Free Transverse Elastic Vibrations of a Solid Cylinder Bonded to a Thin Casing," *Journal SIAM Applied Mathematics*, Vol. 14, No. 2, March 1966, pp. 266–285.